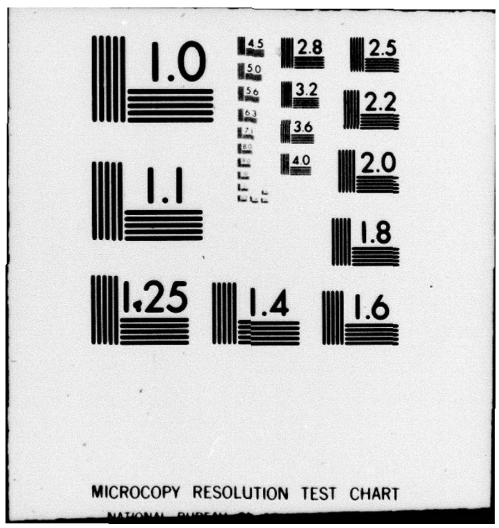
AD-A074 090 MASSACHUSETTS UNIV AMHERST DEPT OF MATHEMATICS AND S--ETC F/G 12/1
ON THE UNIFORMLY MINIMUM VARIANCE UNBIASED ESTIMATORS OF THE VA--ETC(U)
JUL 79 R M KORWAR F49620-79-C-0105
UNCLASSIFIED TR-3 AFOSR-TR-79-0979 NL

| OF |

ADA 074090

END
DATE
FILMED
10 -79
DDC


1.0
1.1
1.25
1.4
1.6
1.8
2.0
2.2
2.5
2.8
3.2
3.6
4.0
4.5
5.0
5.6
6.3
MICROCOPY RESOLUTION TEST CHART

(18) (19)

AFOSR-TR- 79-0979

LEVEL II

(2)

(9) Interim rept.,

(6)

ON THE UNIFORMLY MINIMUM VARIANCE UNBIASED ESTIMATORS

OF THE VARIANCE AND ITS RECIPROCAL OF

AN INVERSE GAUSSIAN DISTRIBUTION[1]

by

(10) Ramesh M. Korwar

(14) TR-

AFOSR Technical Report No. 3

(16) 2304

(17) A5

(12) 10 p.

(11) July 1979

The University of Massachusetts
Department of Mathematics and Statistics
Amherst, Massachusetts

411377

[1]Research sponsored by the Air Force Office of Scientific Research, AFSC, USAF, under Contract F49620-79-C-0105

(15)

On the uniformly minimum variance unbiased estimators

of the variance and its reciprocal of

an Inverse Gaussian distribution

Abstract

The two-parameter Inverse Gaussian distribution is found to have useful applications in a wide variety of fields. The uniformly minimum variance estimator of its mean is known and is the sample mean. However, no such estimator of the variance is reported in literature. Here the uniformly minimum variance unbiased estimators of the variance and its reciprocal are derived.

KEY WORDS: Inverse Gaussian distribution; reciprocal of the variance; variance; uniformly minimum variance unbiased estimator.

On the uniformly minimum variance unbiased estimators
of the variance and its reciprocal of
an Inverse Gaussian distribution

1. INTRODUCTION

The Inverse Gaussian distribution was introduced by Schrödinger (1915), and has been studied extensively by many authors, pioneered by Tweedie (1945, 1956, 1957a, 1957b), and Wald (1945). The distribution is found to have useful applications in a wide variety of fields, such as biology, economics, medicine, and reliability and life testing. See the recent review article by Folks and Chhikara (1978) and discussion on it.

Using Tweedie's notation the probability density function of an Inverse Gaussian random variable X is given by

$$f(x;\, \mu,\lambda) = (\lambda/2\pi x^3)^{\frac{1}{2}} \exp\{-\lambda(x-\mu)^2/2\mu^2 x\}, \quad x > 0,$$

$$= 0 \quad , \quad \text{otherwise} \tag{1.1}$$

where μ,λ are both positive. As far as estimation of the parameters μ and λ is concerned, it is known that the sample mean is the uniformly minimum variance unbiased estimator of the mean μ of X.

However, no such estimator has been obtained in literature for the variance μ^3/λ . See Folks and Chhikara (1978). Theorem 1 below gives this estimator, and Theorem 2 gives the uniformly minimum variance unbiased estimator of the reciprocal λ/μ^3 of the variance.

2. THE ESTIMATORS AND THEIR DERIVATIONS

Let $X_1,\ldots,X_n (n \geq 2)$ be a random sample form (1.1), and $\bar{X}$ be the sample mean, and $V = \sum_{i=1}^{n} \left(\frac{1}{X_i} - \frac{1}{\bar{X}} \right)$. Then

Theorem 1. The uniformly minimum variance unbiased estimator of the variance of (1.1) is given by for $n \geq 3$

$$T(X_1,\ldots X_n) = n\bar{X}^2 (1 + U) \tag{2.1}$$

where U is given by

$$U = \frac{1}{B\left(\frac{1}{2}, \frac{n-2}{2}\right)} \left\{ \frac{n}{V\bar{X}} \sum_{r=1}^{k} \left(\frac{V\bar{X} + n}{V\bar{X}} \right)^{r-1} B\left(\frac{1}{2}, \frac{n-2-2r}{2}\right) - \left(\frac{V\bar{X} + n}{V\bar{X}} \right)^{\frac{n-4}{2}} \sqrt{\frac{n}{V\bar{X}}}\, W \right\}$$

where $k = (n-3)/2$ if n is odd $= (n-4)/2$ if n even, and

$W = 2\tan^{-1}\sqrt{(V\overline{X})/n}$ if n is even and $= \pi$ if n is odd, and

where $B(p,q)$ is the usual beta integral. And for $n = 2$ the estimator is given

$$T(X_1, X_2) = \frac{2\overline{X}^{\,3}V}{V\overline{X} + 2} \quad .$$

Proof: It is known that the statistic $(\overline{X}, V)$ is complete and sufficient for the family (1.1). The sample variance $S^2 = \sum_1^n (X_i - \overline{X})^2/(n-1)$ is unbiased for the variance $\sigma^2 = \mu^3/\lambda$. To finish the proof for the case $n = 2$, we apply the Lehmann-Scheffé Theorem to S^2 and note that $S^2 = 2\overline{X}^{\,3}V/(V\overline{X} + 2)$.

The proof for the case $n \geq 3$ is however more involved. For an application of the Lehmann-Scheffé Theorem using S^2, we need to compute the conditional expectation of X_1^2 given $(\overline{X}, V)$. The conditional density $h(x_1|\overline{x}, v)$ of X_1 given $(\overline{X}, V)$ is derived in Chhikara and Folks (1974) in a different context. It is given by, for $n \geq 3$,

$$h(x_1|\bar{x},v) = \frac{\sqrt{n}\,(n-1)}{B\left(\frac{1}{2}, \frac{n-2}{2}\right)} \left\{\frac{\bar{x}^3}{vx_1^3\,(n\bar{x} - x_1)^3}\right\}^{\frac{1}{2}} \left\{1 - \frac{n(x_1 - \bar{x})^2}{v\bar{x}x_1\,(n\bar{x}-x_1)}\right\}^{\frac{n-4}{2}} \tag{2.2}$$

where $L < x_1 < U$ and L and U are the two roots of the quadratic equation $n(x_1 - \bar{x})^2 = vx_1\bar{x}(n\bar{x} - x_1)$. Now letting $u = \sqrt{n}(x_1-\bar{x})/\{v\bar{x}x_1\,(n\bar{x} - x_1)\}^{\frac{1}{2}}$ in (2.2) we obtain after some simplifications,

$$E(X_1^2|\bar{x},v) = n\bar{x}^2 - \frac{2(n-1)n\bar{x}^2}{B\left(\frac{1}{2}, \frac{n-2}{2}\right)}\, I \tag{2.3}$$

where

$$I = \int_0^1 \frac{1}{n+v\bar{x}u^2}\,(1-u^2)^{\frac{n-4}{2}}\,du \quad . \tag{2.4}$$

To evaluate the integral I in (2.4), first write the integrand as

$$-\frac{1}{v\bar{x}}\sum_{r=1}^{k}\left(\frac{v\bar{x}+n}{v\bar{x}}\right)^{r-1}(1-u^2)^{k-r}+\left(\frac{v\bar{x}+n}{v\bar{x}}\right)^{k}\frac{1}{n+v\bar{x}u^2}\text{ ,n even}$$

$$(2.5)$$

$$-\frac{1}{v\bar{x}}\sum_{r=1}^{k}\left(\frac{v\bar{x}+n}{v\bar{x}}\right)^{r-1}(1-u^2)^{k-r-\frac{1}{2}}+\left(\frac{v\bar{x}+n}{v\bar{x}}\right)^{k}\frac{(1-u^2)^{-\frac{1}{2}}}{n+v\bar{x}u^2}\text{ , n odd}$$

where $k = (n-4)/2$ if n is even and $= (n-3)/2$ if n is odd. (In deriving (2.3) use is made of the fact that $x_1 = n\bar{x}(2+v\bar{x}u^2)/2(n+v\bar{x}u^2) + g(u)$, where g is an odd function of u.) Now to finish the proof use (2.5) in (2.4) and standard integrals, and note that $E(S^2|\bar{x},v) = n\,E(X_1^2|\bar{x},v)/(n-1) - n^2\bar{x}^2/(n-1)$.

Theorem 2: The uniformly minimum variance unbiased estimator of the reciprocal of the variance of (1) is given by, for $n \geq 4$,

$$T(X_1,\ldots,X_n) = \frac{(n-3)}{V\bar{X}^3} - \frac{6}{n\bar{X}^2} + \frac{3}{n^2\bar{X}}\frac{V}{(n-1)}. \qquad (2.6)$$

Proof: The unbiasedness of (2.6) follows from the expressions for negative moments about zero of an Inverse Gaussian distribution given by Tweedie (1957a) and the facts that (i) $\bar{X}$ and V are independent and that (ii) $\bar{X}$ has an Inverse Gaussian distribution with parameters $(\mu, n\lambda)$, (iii) λV has a chi-square distribution with $(n-1)$ degrees of freedom. An application of the Lehmann-Scheffé Theorem to (2.6) using the fact that $(\bar{X}, V)$ is complete as well as sufficient finishes the proof.

REFERENCES

Chhikara, R. S., and Folks, J. L. (1974), "Estimation of the Inverse Gaussian Distribution Function", Journal of the American Statistical Association, 69, 250-4.

Folks, J. L., and Chhikara, R. S. (1978), "The Inverse Gaussian Distribution and its Application - a Review", Journal of the Royal Statistical Society, Ser. B, 40, 263-89.

Schrödinger, E. (1915), "Zur Theorie der Fall-und Steig versuche an Teilchen mit Brownscher Bewegung", Physikalische Zeitschrift, 16, 289-95.

Tweedie, M. C. K. (1945), "Inverse Statistical Variates", Nature, 155,453

_______, (1956), "Some Statistical Properties of Inverse Gaussian Distributions", Virginia Journal of Science, 7, 160-5.

_______, (1957a), "Statistical Properties of Inverse Gaussian Distributions I", Annals of Mathematical Statistics, 28, 362-77.

_______, (1957b), "Statistical Properties of Inverse Gaussian Distributions II", Annals of Mathematical Statistics, 28, 696-705.

Wald ,A. (1947), Sequential Analysis, New York: Wiley.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 79-0979	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON THE UNIFORMLY MINIMUM VARIANCE UNBIASED ESTIMATORS OF THE VARIANCE AND ITS RECIPROCAL OF AN INVERSE GAUSSIAN DISTRIBUTION		5. TYPE OF REPORT & PERIOD COVERED Interim
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Ramesh M. Korwar		8. CONTRACT OR GRANT NUMBER(s) F49620-79-C-0105
9. PERFORMING ORGANIZATION NAME AND ADDRESS The University of Massachusetts Department of Mathematics & Statistics Amherst, MA 01003		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 611C2F 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB, Washington, DC 20332		12. REPORT DATE July 1979
		13. NUMBER OF PAGES 9
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Inverse Gaussian distribution
reciprocal of the variance
variance
uniformly minimum unbiased estimator

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The two-parameter Inverse Gaussian distribution is found to have useful applications in a wide variety of fields. The uniformly minimum variance estimator of its mean is known and is the sample mean. However, no such estimator of the variance is reported in literature. Here the uniformly minimum variance unbiased estimators of the variance and its reciprocal are derived.

DD FORM 1 JAN 73 1473

UNCLASSIFIED